Handout 2

STAT5710 Applied Stochastic Processes

Here we consider the long-run behavior of Markov chains. Let $\{X_t\}$ be a time-homogeneous Markov chain on a state space S, with $p_{ij}^{(n)} = (P^n)_{ij}$ and $P(X_n = j) = (\alpha P^n)_j$.

The chain has a **limiting distribution** if $\lim_{n\to\infty} p_{ij}^{(n)}$ exists for each $i, j \in S$ and defines a probability distribution on S for each i. Such a limit is

A. **weak** if $\lim\limits_{n\to\infty} p_{ij}^{(n)} = \pi_{ij}$

B. **strong** if $\lim\limits_{n\to\infty} p_{ij}^{(n)} = \pi_j$

In words, the limit represents the probability of transitioning to state j at a time arbitrarily far into the future; that limit is weak if it depends on the starting state.

The limiting distribution of a two-state chain

Consider a two-state markov chain defined by

$$P = \begin{array}{cc} & \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \end{array}$$

This is possibly the simplest Markov chain we might construct. Here we will derive the limit analytically in this simple case before considering how to find limiting distributions in more general settings.

1. Draw the transition graph.

2. Condition on the state at time $n-1$ to get a recursion for $p_{11}^{(n)}$.

$$p_{11}^{(n)} \;=\; p_{11}^{(n-1)}\,\underline{\qquad}\;+\;(1-p_{11}^{(n-1)})\,\underline{\qquad}\;=\;\underline{\qquad}\;+\;(1-p-q)\,p_{11}^{(n-1)}$$

3. Unrolling the recursion and summing the geometric series gives

$$p_{11}^{(n)} \;=\; \frac{q}{p+q} \;+\; \frac{p}{p+q}\,(1-p-q)^n$$

Find $p_{22}^{(n)}$.

$$p_{22}^{(n)} = \rule{6cm}{0.4pt}$$

4. Now find the remaining entries.

$$p_{12}^{(n)} = 1 - p_{11}^{(n)} = \rule{10cm}{0.4pt}$$

$$p_{21}^{(n)} = 1 - p_{22}^{(n)} = \rule{10cm}{0.4pt}$$

5. Write P^n.

6. Everything about the long run is controlled by the single quantity $(1-p-q)^n$.

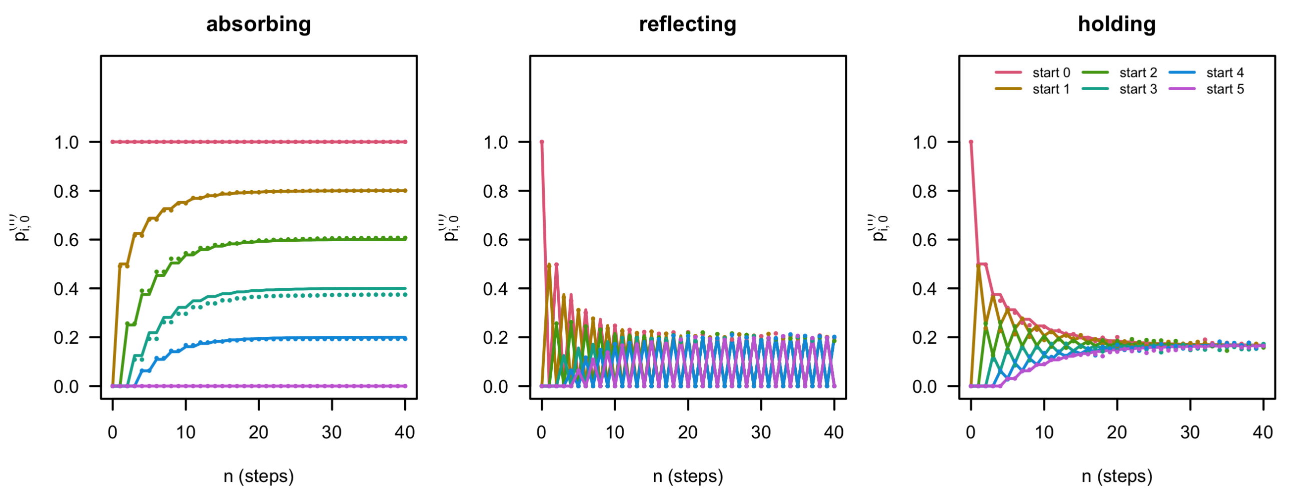
   (a) For which values of p and q is $|1-p-q| < 1$? What is $\lim_{n\to\infty}(1-p-q)^n$ in this case?

   (b) For which values of p and q is $|1-p-q| = 1$? What is $\lim_{n\to\infty}(1-p-q)^n$ in this case?

7. Fill in the table. Below the table, write the limit of P^n in each case where it exists.

(p, q)	weak / strong / no limit
$p = .15$, $q = .25$	
$p = q = 0$	
$p = q = 1$	

8. Look at the Gambler's ruin simulations from earlier; the plots are reproduced below. Characterize each case according to whether a limit exists and if so, what type of limit.



From limits to marginals

A limiting distribution says something about $p_{ij}^{(n)}$, which conditions on where the chain started. The marginal $P(X_n = j)$ does not. The two are connected.

Lemma. If a limiting distribution exists, then $\lim_n P(X_n = j)$ exists for every initial distribution α.

9. Complete the proof. Write $\alpha_k = P(X_0 = k)$ and $\pi_{kj} = \lim_n p_{kj}^{(n)}$.

$$P(X_n = j) \ = \ \sum_k \underline{\hspace{6em}} \cdot \underline{\hspace{4em}} \ \longrightarrow \ \sum_k \underline{\hspace{6em}} \cdot \underline{\hspace{4em}} \qquad \text{as } n \to \infty$$

10. Now specialize, and notice that this is where weak and strong limits behave differently.

(a) If the limit is **strong**, then $\pi_{kj} = \pi_j$ for every k. Show that $\lim_n P(X_n = j) = \pi_j$. Does this depend on α?

(b) If the limit is **weak**, what is $\lim_n P(X_n = j)$? Does it depend on α?

Stationary distributions

A limit is not the only way the long run can be well behaved. Take the oscillating chain, $p = q = 1$, which has no limiting distribution at all, and start it at $\alpha^T = (\frac{1}{2}, \frac{1}{2})$.

11. Compute $\alpha^T P$.

12. Use that to show $P(X_n = j)$ is the *same for every* n.

$$\begin{aligned} P(X_n = j) &= \alpha^T P^n \\ &= \alpha^T (P P^{n-1}) \\ &= (\underline{\hspace{3cm}}) P^{n-1} \\ &= \underline{\hspace{2cm}} P^{n-1} \\ &\vdots \\ &= \underline{\hspace{2cm}} \end{aligned}$$

A distribution π is a **stationary distribution** if

$$\sum_k \pi_k \, p_{kj} = \pi_j \quad \text{for every } j, \qquad \text{equivalently (finite } S) \qquad \pi^T P = \pi^T$$

Lemma. If π is a limiting distribution then it is also a stationary distribution.

13. Complete the proof by splitting off one step at the end.

$$\pi_{ij} \;=\; \lim_n p_{ij}^{(n)} \;=\; \lim_n \sum_k \underline{\hspace{2.5cm}} \cdot \underline{\hspace{1.5cm}} \;=\; \sum_k \underline{\hspace{2.5cm}} \cdot \underline{\hspace{1.5cm}}$$

14. The converse fails. Use the $p = q = 1$ chain to explain why, in one sentence.

Two questions to carry forward. When does a stationary distribution exist? When does a limiting distribution exist?

Existence when S is finite

Lemma. If S is finite then a stationary distribution exists.

The proof is short, but it rests on some linear algebra. Fill in the review as we go.

15. **Eigenvalue review.** λ is an **eigenvalue** of a square matrix A if ______________ for some $v \neq 0$
    (a) Such a v is a ____________________________
    (b) Rewriting the condition as $(A - \lambda I)v = 0$, a nonzero solution exists when…
    (c) So the eigenvalues are the roots of the **characteristic polynomial**
    $$p(\lambda) = \underline{\hspace{4cm}}$$
    (d) An $m \times m$ matrix has at most _____ distinct eigenvalues.
    (e) The **algebraic multiplicity** is k if $p(t) \propto$ ______________
    (f) The **geometric multiplicity** is q if…
    (g) A and A^T have the same eigenvalues because
    $$\det(A^T - \lambda I) = \det\left[(A - \lambda I)^T\right] = \det(A^T - \lambda I) \quad \Longrightarrow \quad \underline{\hspace{5cm}}$$
    (h) We say u is a **left eigenvector** of A for λ if…
    (i) The **spectral radius** of A is defined as $\rho(A) =$ _________________

Theorem (Perron–Frobenius). If A is nonnegative, meaning $a_{ij} \geq 0$ for every i, j, then $\rho(A)$ is an eigenvalue of A, and there is a corresponding nonnegative right eigenvector.

16. **Proof.** Prove the lemma in three steps.

   (a) Let $\mathbf{1}$ be the vector of ones. Compute $P\mathbf{1}$ using the fact that P is stochastic, and say what eigenvalue you have just found.

   (b) Now show no eigenvalue can be larger in absolute value. Using $\|x\|_\infty = \max_i |x_i|$, we have $\|Px\|_\infty \le \|x\|_\infty$ for any x. Apply this to an eigenvector v with eigenvalue λ.

   $$\|Pv\|_\infty \le \|v\|_\infty \qquad \Longrightarrow \qquad \underline{\hspace{3cm}} \qquad \Longrightarrow \qquad \underline{\hspace{3cm}}$$

   Explain why the first inequality is true.

   (c) By 15(g), $\rho(P^T) = 1$ as well. Apply Perron–Frobenius to P^T and explain why the resulting vector, after normalizing, is a stationary distribution.

17. The proof shows that for a finite chain, the stationary distributions are

18. The stationary distribution is unique when…