1. Consider the two-state homogeneous Markov chain $X_n \in \{0, 1\}$ with transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

and let $\alpha = (\alpha_0, \alpha_1)$ denote the initial distribution of the chain.

(a) Find the two-step transition probabilities (i.e., compute P^2).

(b) Find the marginal expectation $E[X_2]$.

(c) Show that $E[X_n] = P(X_n = 1)$.

(d) In class we derived a general expression for the n-step transition matrix P^n. Use this to find $E[X_n]$.

(e) Compute and interpret $\lim_{n\to\infty} E[X_n]$ (assume $p + q > 0$).

(f) Let $T_1 = \min\{n > 0 : X_n = 1\}$ be the first passage time to state 1. Compute $E[T_1 \mid X_0 = 1]$ in terms of p and q, and give a brief interpretation.

Solution

(a) $P^2 = \begin{bmatrix} (1-p)^2 + pq & p(2-p-q) \\ q(2-p-q) & (1-q)^2 + pq \end{bmatrix}$

(b) $E[X_2] = P(X_2 = 1) = (\alpha P^2)_1 = \alpha_0 (P^2)_{01} + \alpha_1 (P^2)_{11} = \alpha_0\, p(2-p-q) + \alpha_1 \big[(1-q)^2 + pq\big]$

(c) $E[X_n] = 0 \cdot P(X_n = 0) + 1 \cdot P(X_n = 1) = P(X_n = 1)$

(d) In class we derived

$$P^n = \frac{1}{p+q} \begin{bmatrix} q + pc_n & p - pc_n \\ q - qc_n & p + qc_n \end{bmatrix} \qquad \text{where} \qquad c_n = (1-p-q)^n$$

Hence

$$E[X_n] = P(X_n = 1) = (\alpha P^n)_1 = \alpha_0 \frac{p - pc_n}{p+q} + \alpha_1 \frac{p + qc_n}{p+q} = \frac{p}{p+q} + \Big(\alpha_1 - \frac{p}{p+q}\Big) c_n$$

(e) Using the result from the previous part, consider two cases.

i. If $p + q < 2$ then $c_n \to 0$, so $\lim_{n\to\infty} E[X_n] = \frac{p}{p+q}$.

ii. If $p + q = 2$ then $c_n = (-1)^n$ and $E[X_n] = \frac{1}{2} + (\alpha_1 - \frac{1}{2})(-1)^n$. The sequence of expectations has no limit unless $\alpha_1 = \frac{1}{2}$, in which case $E[X_n] \equiv \frac{1}{2}$.

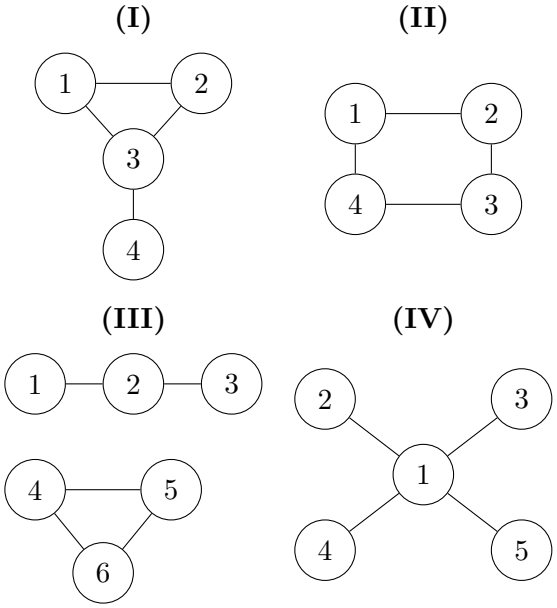
When the limit exists, it gives the long-run probability the chain is in state 1, and therefore the long-run fraction of time the chain spends in state 1.

(f) Continuing to assume $p, q > 0$, the chain has a unique stationary distribution (derived in class):

$$\pi = \Big(\frac{q}{p+q},\ \frac{p}{p+q}\Big)$$

Then $E[T_1 \mid X_0 = 1] = \pi_1^{-1} = \frac{p+q}{p}$ gives the expected return time to state 1, i.e., the expected number of steps between successive visits.

2. For each undirected graph (I)–(IV) below, consider the simple random walk: from a vertex, move at each step to a uniformly chosen neighbor. That is, $P_{ij} = \frac{1}{\deg(i)}$ if vertices i and j are adjacent, and $P_{ij} = 0$ otherwise.



(a) For graph (I) write the transition matrix P.

(b) For each chain, identify the communication classes.

(c) Classify each communication class as transient or recurrent.

(d) Determine the period of each communication class.

(e) Which of the four chains are ergodic? Briefly justify.

(f) Repeat (a)-(e) if the random walks are *lazy*: from any state, the chain moves with uniform probability to a neighbor or itself.

Solution

(a) Graph (I): degrees are $\deg(1) = 2$, $\deg(2) = 2$, $\deg(3) = 3$, $\deg(4) = 1$, so

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Communication classes: (I) $\{1, 2, 3, 4\}$; (II) $\{1, 2, 3, 4\}$; (III) $\{1, 2, 3\}$ and $\{4, 5, 6\}$; (IV) $\{1, 2, 3, 4, 5\}$.

(c) All classes are recurrent.

(d) Periods: (I) aperiodic; (II) period 2; (III) $\{1, 2, 3\}$ has period 2, $\{4, 5, 6\}$ is aperiodic; (IV) period 2.

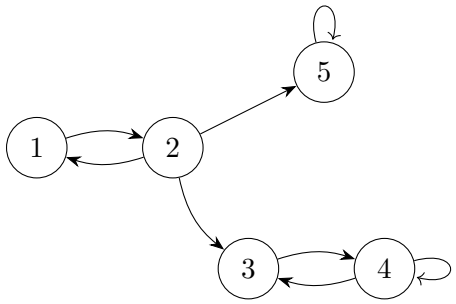
(e) Ergodic $\iff$ irreducible and aperiodic. Only (I) is ergodic. (II) and (IV) are periodic; (III) is reducible.

(f) Now each state keeps a self-loop, so from state i the walk chooses uniformly among $\deg(i)+1$ options and $p_{ii} = p_{ij} = \frac{1}{\deg(i)+1}$ for each neighbor j. For graph (I) the denominators are $3, 3, 4, 2$:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Self-loops do not change which states communicate, so the classes in (b) and their recurrence in (c) are unchanged. But every state now has $p_{ii} > 0$, so every class is aperiodic. Hence (I), (II), and (IV) are ergodic; (III) is still reducible and so is not, though each of its two classes is now aperiodic and recurrent.

3. Consider the Markov chain on states $\{1, 2, 3, 4, 5\}$ whose transitions are given by the directed graph below. At each step, the chain chooses uniformly among outgoing edges (self-loops count as outgoing edges).



(a) Write the transition matrix P in canonical form.

(b) Compute the fundamental matrix $F = (I - Q)^{-1}$ and interpret each entry.

(c) Starting from state 1, how many steps are expected until the chain enters a closed class?

(d) Starting from state 1, what is the probability that the chain is absorbed into state 5?

(e) What is the limiting distribution (if it exists)?

Solution

Use canonical order $(1, 2 \mid 3, 4, 5)$, transient states first.

(a) $Q = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{bmatrix}$, $\quad R = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$, $\quad S = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) F_{ij} is the expected number of visits to state j starting from state i, before absorption into a closed class.

$$F = (I - Q)^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

(c) Expected time to absorption starting from state 1 is $(F\mathbf{1})_1 = 3$.

(d) Absorption probabilities into the closed classes from each state are

$$FR = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

so $P(\text{absorb in } 5 \mid X_0 = 1) = \frac{1}{2}$.

(e) In class we showed that when the closed classes are aperiodic,

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & (I - Q)^{-1} R\, \Pi \\ 0 & \Pi \end{bmatrix}$$

where $\Pi = \lim_{n\to\infty} S^n$ is constructed as a block-diagonal matrix from the stationary distribution(s) of the closed class(es). In this case, the closed classes are $\{3, 4\}$ and $\{5\}$.

On $\{3,4\}$ the stationary distribution is $(\frac{1}{3},\frac{2}{3})$ and state 5 is absorbing, so

$$\Pi = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$(I-Q)^{-1}R = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad \Longrightarrow \qquad (I-Q)^{-1}R\,\Pi = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

And

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. A weather station records the prevailing wind direction once every hour. Let $X_n \in \{N, E, S, W\}$ denote the wind direction during hour n. Data spanning a 96-hour period yielded the following transition counts:

	N	E	S	W
N	18	6	2	4
E	5	15	6	4
S	1	3	0	1
W	6	4	3	17

   (a) Compute the MLE $\hat{P}$ of the transition matrix.

   (b) Explain briefly why the MLE may be unreliable here for row S.

   (c) Use Laplace smoothing to adjust the estimate in (a) for row S only.

   (d) Use Dirichlet smoothing to adjust the estimate in (a) for row S only with an empirical Bayes prior: let the prior mean be the pooled transition proportions from remaining rows (column margin after removing row S), scaled by a factor of $\alpha = 10$ (strength).

Solution

   (a) The MLE is obtained by row normalization:

$$\hat{P} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{15} & \frac{2}{15} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{5} & \frac{2}{15} \\ \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{15} & \frac{1}{10} & \frac{17}{30} \end{bmatrix}$$

   (b) Row S has only 5 transitions (small sample size) and includes a zero count ($S \to S$), so (a) there is high uncertainty for that row and (b) the estimate implies the wind never stays south.

(c) Laplace smoothing on row S gives

$$(1,3,0,1) \mapsto (2,4,1,2) \qquad \Longrightarrow \qquad \tilde{p}_S = \left(\tfrac{2}{9}, \tfrac{4}{9}, \tfrac{1}{9}, \tfrac{2}{9}\right)$$

(d) Column totals from rows N, E, W are

$$(N, E, S, W) = (29, 25, 11, 25)$$

The empirical Bayes prior is then

$$\left(\tfrac{29}{9}, \tfrac{25}{9}, \tfrac{11}{9}, \tfrac{25}{9}\right)$$

yielding

$$\tilde{p}_S = \left(\tfrac{38}{135}, \tfrac{52}{135}, \tfrac{11}{135}, \tfrac{34}{135}\right)$$