This practice set is about twice the length of the exam, which is roughly nine parts in 80 minutes. Use it to check coverage, not pace.

1. Consider the two-state homogeneous Markov chain $X_n \in \{0, 1\}$ with transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

and let $\alpha = (\alpha_0, \alpha_1)$ denote the initial distribution of the chain.

   (a) Find the two-step transition probabilities (i.e., compute P^2).

   (b) Find the marginal expectation $E[X_2]$.

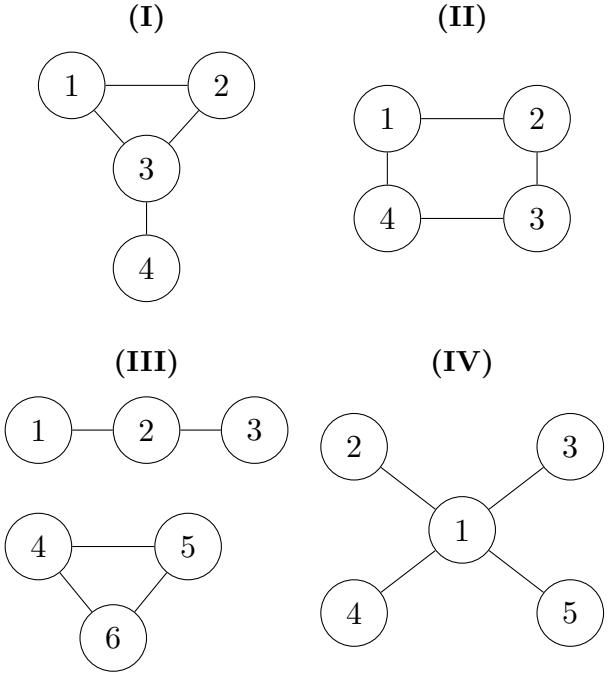
   (c) Show that $E[X_n] = P(X_n = 1)$.

   (d) In class we derived a general expression for the n-step transition matrix P^n. Use this to find $E[X_n]$.

   (e) Compute and interpret $\lim_{n\to\infty} E[X_n]$ (assume $p + q > 0$).

   (f) Let $T_1 = \min\{n > 0 : X_n = 1\}$ be the first passage time to state 1. Compute $E[T_1 \mid X_0 = 1]$ in terms of p and q, and give a brief interpretation.

2. For each undirected graph (I)–(IV) below, consider the simple random walk: from a vertex, move at each step to a uniformly chosen neighbor. That is, $P_{ij} = \frac{1}{\deg(i)}$ if vertices i and j are adjacent, and $P_{ij} = 0$ otherwise.



(a) For graph (I) write the transition matrix P.

(b) For each chain, identify the communication classes.

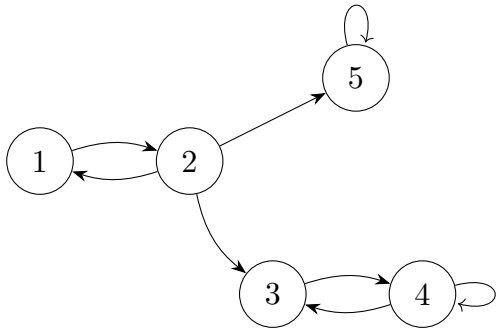
(c) Classify each communication class as transient or recurrent.

(d) Determine the period of each communication class.

(e) Which of the four chains are ergodic? Briefly justify.

(f) Repeat (a)-(e) if the random walks are *lazy*: from any state, the chain moves with uniform probability to a neighbor or itself.

3. Consider the Markov chain on states $\{1, 2, 3, 4, 5\}$ whose transitions are given by the directed graph below. At each step, the chain chooses uniformly among outgoing edges (self-loops count as outgoing edges).



   (a) Write the transition matrix P in canonical form.

   (b) Compute the fundamental matrix $F = (I - Q)^{-1}$ and interpret each entry.

   (c) Starting from state 1, how many steps are expected until the chain enters a closed class?

   (d) Starting from state 1, what is the probability that the chain is absorbed into state 5?

   (e) What is the limiting distribution (if it exists)?

4. A weather station records the prevailing wind direction once every hour. Let $X_n \in \{N, E, S, W\}$ denote the wind direction during hour n, and suppose hourly changes in wind direction can be modeled as a time-homogeneous Markov chain. Data spanning a 96-hour period yielded the following transition counts:

	N	E	S	W
N	18	6	2	4
E	5	15	6	4
S	1	3	0	1
W	6	4	3	17

   (a) Compute the MLE $\hat{P}$ of the transition matrix.

   (b) Explain briefly why the MLE may be unreliable here for row S.

   (c) Use Laplace smoothing to adjust the estimate in (a) for row S only.

   (d) Use Dirichlet smoothing to adjust the estimate in (a) for row S only with an empirical Bayes prior: let the prior mean be the pooled transition proportions from remaining rows (column margin after removing row S), scaled by a factor of $\alpha = 10$ (strength).